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SIMPLER STATIC PROBLEMS IN NONLINEAR THEORIES OF RODS

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Abstract—In many nonlinear theories of rods, certain inverse assumptions permit replacement of the governing differential equations by algebraic relations. For a rather general abstract theory of rods, summarized below, we illustrate how to accomplish this.

1. PRELIMINARIES

WE CONSIDER elastostatic theories of rods similar to those proposed by Cohen [1] and Green and Laws [2], beginning with a summary of relevant features. Abstractly, a rod is a curve equipped with two vectors. The curve is given parametrically by

$$\mathbf{r} = \mathbf{r}(S), \tag{1}$$

$\mathbf{r}$ being the position vector from some origin and S a material coordinate. The two vectors, denoted by

$$\mathbf{d}_\alpha = \mathbf{d}_\alpha(S), \qquad \alpha = 1, 2, \tag{2}$$

and the tangent vector

$$\mathbf{r}'(S) \tag{3}$$

are to form a linearly independent set. Throughout, Greek indices take on values 1 and 2 and the summation convention applies to them. Further, primes denote derivatives with respect to S. We impose no other constraints on these vectors, though the analysis is easily modified to allow for constraints such as are imposed in classical theories. Essentially the same analysis applies if there are more than two vectors. Cohen [1] introduces three.

The stored energy associated with the element of material between S and $S+\mathrm{d}S$ is written

$$W\,\mathrm{d}S,$$

where

$$W = W(S, \mathbf{r}', \mathbf{r}'', \mathbf{d}_\alpha, \mathbf{d}'_\alpha) \tag{4}$$

is an objective scalar. That is, if $\mathbf{U}$ represents any rigid rotation,

$$\mathbf{U}^{-1} = \mathbf{U}^T, \qquad \det \mathbf{U} = 1, \tag{5}$$

and if bars denote the transforms of vector arguments by $\mathbf{U}$, e.g.

$$\bar{\mathbf{r}}' = \mathbf{U}\mathbf{r}', \qquad \bar{\mathbf{r}}'' = \mathbf{U}\bar{\mathbf{r}}'', \ldots, \tag{6}$$

then

$$W(S, \bar{\mathbf{r}}', \bar{\mathbf{r}}'', \bar{\mathbf{d}}_\alpha, \bar{\mathbf{d}}_\alpha') = W(S, \mathbf{r}', \mathbf{r}'', \mathbf{d}_\alpha, \mathbf{d}_\alpha). \tag{7}$$

As a consequence, derivatives of W with respect to vectors transform as vectors, e.g.

$$\frac{\partial W}{\partial \bar{\mathbf{r}}''} = \mathbf{U}\frac{\partial W}{\partial \mathbf{r}''}. \tag{8}$$

Further, (7) implies that W satisfies the identity

$$\mathbf{r}' \times \frac{\partial W}{\partial \mathbf{r}'} + \mathbf{r}'' \times \frac{\partial W}{\partial \mathbf{r}''} + \mathbf{d}_\alpha \times \frac{\partial W}{\partial \mathbf{d}_\alpha} + \mathbf{d}_\alpha' \times \frac{\partial W}{\partial \mathbf{d}_\alpha'} \equiv 0. \tag{9}$$

Though we make no use of it, W is expressible in terms of the scalar and triple scalar products formed from the vector arguments. By a straightforward calculation,

$$\delta \int_{S_1}^{S_2} W\,\mathrm{d}S = (\mathbf{F}\cdot\delta\mathbf{r} + \mathbf{G}\cdot\delta\mathbf{r}' + \mathbf{H}^\alpha\cdot\delta\mathbf{d}_\alpha)\Big|_{S_1}^{S_2} + \int_{S_1}^{S_2} (\mathbf{K}\cdot\delta\mathbf{r} + \mathbf{L}^\alpha\cdot\delta\mathbf{d}_\alpha)\,\mathrm{d}S, \tag{10}$$

where S_1 and S_2 are any two constants, and

$$\begin{aligned}
\mathbf{F} &\equiv \frac{\partial W}{\partial \mathbf{r}'} - \mathbf{G}', \\
\mathbf{G} &\equiv \frac{\partial W}{\partial \mathbf{r}''}, \\
\mathbf{H}^\alpha &\equiv \frac{\partial W}{\partial \mathbf{d}_\alpha'}, \\
\mathbf{K} &\equiv -\mathbf{F}', \\
\mathbf{L}^\alpha &\equiv \frac{\partial W}{\partial \mathbf{d}_\alpha} - \left(\frac{\partial W}{\partial \mathbf{d}_\alpha'}\right)'.
\end{aligned} \tag{11}$$

Equilibrium equations are obtained by equating $\mathbf{K}$ and $\mathbf{L}^\alpha$ to specified functions, representing generalized forces applied along the rod. For specifications of the type

$$\mathbf{K} = 0, \qquad \mathbf{L}^\alpha = -\frac{\partial \Phi}{\partial \mathbf{d}_\alpha},$$

where

$$\Phi = \Phi(\mathbf{d}_\alpha),$$

we can of course eliminate the integral term in the right member of (9), if we replace W by $W+\Phi$. If Φ is objective, the modified energy has the same properties as Φ and our analysis would apply. For simplicity, we restrict our attention to cases where these forces vanish,

$$\mathbf{K} = \mathbf{L}_\alpha = 0. \tag{12}$$

Then, using (11),

$$\mathbf{F} = \frac{\partial W}{\partial \mathbf{r}'} - \left(\frac{\partial W}{\partial \mathbf{r}''}\right)' = \mathbf{a} = \text{const}. \tag{13}$$

This integral of the equilibrium equations represents the requirement that the resultant force on any section of the rod must vanish. Balance of moments gives rise to another integral. To obtain it, consider a variation corresponding to an infinitesimal rigid motion

$$\delta\mathbf{r} = \boldsymbol{\omega}\times\mathbf{r}, \qquad \delta\mathbf{d}_\alpha = \boldsymbol{\omega}\times\mathbf{d}_\alpha,$$

$\boldsymbol{\omega}$ being an arbitrary constant vector. Because of objectivity, the left member of (10) vanishes, so we obtain

$$\mathbf{r}\times\mathbf{F}+\mathbf{r}'\times\mathbf{G}+\mathbf{d}_\alpha\times\mathbf{H}^\alpha = \mathbf{b} = \text{const.}, \tag{14}$$

assuming (12) holds. One more integral obtains in cases where W does not depend explicitly on S. To obtain it set

$$\delta\mathbf{r} = \mathbf{r}', \qquad \delta\mathbf{d}_\alpha = \mathbf{d}'_\alpha$$

in (10), use

$$\delta\int_{S_1}^{S_2} W\,\mathrm{d}S = \int_{S_1}^{S_2} W'\,\mathrm{d}S = W\Big|_{S_1}^{S_2},$$

which gives

$$W-\mathbf{F}\cdot\mathbf{r}'-\mathbf{G}\cdot\mathbf{r}''-\mathbf{H}^\alpha\cdot\mathbf{d}'_\alpha = a = \text{const.} \tag{15}$$

This independency, as well as the form of W are not preserved under nonlinear transformations of the material coordinate S, so an integral similar to (15) can be obtained in some cases where W depends explicitly on S.

That these integrals are satisfied does not imply that the equilibrium equations hold. Of course, $\mathbf{K} = 0$ when (13) applies. Differentiating (14) and using (13) and (9), we obtain the consequence

$$\mathbf{d}_\alpha\times\mathbf{L}^\alpha = 0.$$

Dotting this with $\mathbf{d}_1$ and $\mathbf{d}_2$, we see that

$$\mathbf{d}_1\cdot\mathbf{d}_2\times\mathbf{L}^2 = \mathbf{d}_2\cdot\mathbf{d}_1\times\mathbf{L}^1 = 0,$$

so $\mathbf{L}^\alpha$ lies in the plane determined by $\mathbf{d}_1$ and $\mathbf{d}_2$. It then follows easily that there exist scalars α, β and γ such that

$$\begin{aligned}\mathbf{L}^1 &= \alpha\mathbf{d}_1+\beta\mathbf{d}_2,\\ \mathbf{L}^2 &= \beta\mathbf{d}_1+\gamma\mathbf{d}_2.\end{aligned} \tag{16}$$

Thus, to satisfy the equilibrium equations, it is necessary to add conditions implying that these scalars vanish. Intuitively, these conditions imply balance of resultant force and moment, but the generalized forces include also double forces without moment, which must also be appropriately balanced. If we add constraints requiring $\mathbf{d}_1$ and $\mathbf{d}_2$ to be of fixed magnitude and to include a fixed angle, equilibrium equations are of the form (16), the scalars being then Lagrange multipliers, so no further conditions are required. Generally, (15) yields an independent consequence which can be reduced to the form

$$2\alpha(\mathbf{d}_1\cdot\mathbf{d}_1)'+\beta(\mathbf{d}_1\cdot\mathbf{d}_2)'+2\gamma(\mathbf{d}_2\cdot\mathbf{d}_2)' = 0. \tag{17}$$

2. UNIFORM STATES

The vector arguments upon which W depends serve to define the local state. Because of objectivity, two states which differ by a rotation are essentially the same. Thus, we say that a rod is in a *uniform state* when the state at one point differs from that at any other by such a rotation. Formally

$$\begin{aligned} \mathbf{r}' &= \mathbf{U}\mathbf{R}', \\ \mathbf{r}'' &= \mathbf{U}\mathbf{R}'', \\ \mathbf{d}_\alpha &= \mathbf{U}\mathbf{D}_\alpha, \\ \mathbf{d}'_\alpha &= \mathbf{U}\mathbf{D}'_\alpha, \end{aligned} \tag{18}$$

where $\mathbf{U}$ satisfies (5) and capital letters denote values at a fixed point, say $S = 0$. Here $\mathbf{U}$ may depend on S, subject to the condition that

$$\mathbf{U} = \mathbf{1} \quad \text{when} \quad S = 0. \tag{19}$$

We set

$$\mathbf{\Omega} \equiv \mathbf{U}'\mathbf{U}^{-1} = -\mathbf{\Omega}^T \tag{20}$$

and denote by $\boldsymbol{\omega}$ the corresponding axial vector such that, for any vector $\mathbf{v}$,

$$\mathbf{\Omega}\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}. \tag{21}$$

For consistency,

$$\begin{aligned} \mathbf{r}'' &= (\mathbf{U}\mathbf{R}')' = \mathbf{\Omega}\mathbf{U}\mathbf{R}' \\ &= \mathbf{U}\mathbf{R}'' = \mathbf{U}\mathbf{\Omega}_0\mathbf{R}', \end{aligned} \tag{22}$$

where

$$\mathbf{\Omega}_0 = \mathbf{\Omega}|_{S=0}.$$

Thus

$$(\mathbf{\Omega}\mathbf{U} - \mathbf{U}\mathbf{\Omega}_0)\mathbf{R}' = 0.$$

Similarly

$$(\mathbf{\Omega}\mathbf{U} - \mathbf{U}\mathbf{\Omega}_0)\mathbf{D}_\alpha = 0.$$

Thus, $\mathbf{R}'$ and $\mathbf{D}_\alpha$ being linearly independent,

$$\mathbf{\Omega} = \mathbf{U}\mathbf{\Omega}_0\mathbf{U}^T,$$

or, equivalently,

$$\boldsymbol{\omega} = \mathbf{U}\boldsymbol{\omega}_0.$$

Thus

$$\boldsymbol{\omega}' = \mathbf{\Omega}\mathbf{U}\boldsymbol{\omega}_0 = \mathbf{\Omega}\boldsymbol{\omega} = \boldsymbol{\omega} \times \boldsymbol{\omega} = 0.$$

Thus $\boldsymbol{\omega}$ is a constant vector, such that

$$\mathbf{U}\boldsymbol{\omega} = \boldsymbol{\omega}. \tag{23}$$

From the well-known properties of rigid motions with constant angular velocity, we can prescribe an arbitrary constant vector $\boldsymbol{\omega}$ and determine the corresponding rotation $\mathbf{U}$, which, using (19), is uniquely determined by $\boldsymbol{\omega}$. It will satisfy (23).

Now, from (18) and (22)

$$\mathbf{r}'' = \boldsymbol{\Omega}\mathbf{r}' = \boldsymbol{\omega} \times \mathbf{r}',$$

which can be integrated to give

$$\mathbf{r}' - \boldsymbol{\omega} \times \mathbf{r} = \mathbf{c} = \text{constant}. \tag{24}$$

There is no loss in generality in assuming

$$\mathbf{c} = b\boldsymbol{\omega}, \tag{25}$$

which results from suitable adjustment of the point from which the position vector originates. One more integration gives, for the curve,

$$\begin{aligned} \mathbf{r} &= \mathbf{U}\mathbf{R} + b\boldsymbol{\omega}S, \\ \mathbf{R} &= \mathbf{r}|_{S=0}. \end{aligned} \tag{26}$$

As is clear from (18), S differs from actual arc length on these curves by at most a linear transformation. Along with (26), we of course have

$$\mathbf{d}_\alpha = \mathbf{U}\mathbf{D}_\alpha. \tag{27}$$

Generally, the curves are circular helices, $\boldsymbol{\omega}$ being parallel to the axis of the circular cylinder on which they lie. One degenerate case, with $b = 0$, consists of circles, $\boldsymbol{\omega}$ being perpendicular to their planes. Another involves straight lines, parallel to $\boldsymbol{\omega}$, including the still more degenerate case of straight lines with $\boldsymbol{\omega} = 0$. As is perhaps obvious, the $\mathbf{d}_\alpha$ form constant angles with the principal normal and binormal vectors of the curves, except for the straight lines where these normals are not uniquely defined.

This provides a rather detailed characterization of uniform states, insofar as their kinematics is concerned.

3. ALGEBRAIC SOLUTION

We now restrict our attention to cases where W does not depend explicitly on S. Among other things, this implies that W is constant in every uniform state. Intuitively, a rod which, when unloaded is, say, of variable curvature, is likely to require more work to straighten the more bent parts. Generally, the assumption is not likely to be applicable unless the unloaded rod can reasonably be considered to be in a uniform state. Even granting this, there remain a variety of possibilities for deforming a rod from one uniform state to another. Even the case where the final state is a straight line with constant $\mathbf{d}_\alpha$ can be non-trivial if the initial state were a helical spring or a spiral column. However, for simplicity, we assume that

$$\boldsymbol{\omega} \neq 0. \tag{28}$$

Cohen [1] discusses a case where $\boldsymbol{\omega} = 0$. In what follows, $\mathbf{U}$ refers to the particular rotation associated with the uniform state considered. From (8) and (18) it follows that, in obvious notation, we have relations of the type

$$\frac{\partial W}{\partial \mathbf{r}''} = \mathbf{U}\frac{\partial W}{\partial \mathbf{R}''}, \tag{29}$$

so that

$$\left(\frac{\partial W}{\partial \mathbf{r}''}\right)' = \boldsymbol{\omega} \times \frac{\partial W}{\partial \mathbf{r}''}. \tag{30}$$

Thus

$$\mathbf{F} = \frac{\partial W}{\partial \mathbf{r}'} - \boldsymbol{\omega} \times \frac{\partial W}{\partial \mathbf{r}''} = \mathbf{U}\mathbf{F}_0, \tag{31}$$

where the subscript zero indicates evaluation at $S = 0$. Similarly

$$\mathbf{L}^\alpha = \frac{\partial W}{\partial \mathbf{d}_\alpha} - \boldsymbol{\omega} \times \frac{\partial W}{\partial \mathbf{d}'_\alpha} = \mathbf{U}\mathbf{L}_0^\alpha. \tag{32}$$

In effect, (31) and (32) characterize the generalized forces required to maintain the uniform state.

If it is to be maintained by end loads alone, (13) and (31) require that

$$\mathbf{U}\mathbf{F}_0 = \mathbf{F}_0. \tag{33}$$

Since a rigid rotation changes every direction but its axis, this implies that, for some constant scalar c,

$$\mathbf{F}_0 = c\boldsymbol{\omega}. \tag{34}$$

Clearly, (28) is important here. In the exceptional case where (28) fails, $\mathbf{U} = 1$ and $\mathbf{F}_0$ is unrestricted. If we arrange that (34) holds and

$$\mathbf{L}_0^\alpha = 0, \tag{35}$$

(31) and (32) will imply that the equilibrium equations (12) hold. The initial state is subject to kinematic restrictions, viz.

$$\mathbf{R}'' = \boldsymbol{\omega} \times \mathbf{R}',$$
$$\mathbf{D}'_\alpha = \boldsymbol{\omega} \times \mathbf{D}_\alpha,$$

being thus determined by the three vectors $\mathbf{R}'$, $\mathbf{D}_\alpha$ and $\boldsymbol{\omega}$, which must be selected so as to satisfy (34) and (35) for some value of the constant c. Thinking of c as given, (34) and (35) give three vector equations for the three vectors $\mathbf{R}'$, $\mathbf{D}_\alpha$ and $\boldsymbol{\omega}$. We can not expect these conditions to uniquely determine the vectors, objectivity implying that, at best, they are determined only to within a rigid rotation which, for the latter alternative, must have $\boldsymbol{\omega}$ as axis. Alternatively, we can specify $\boldsymbol{\omega}$, calculating $\mathbf{R}'$, $\mathbf{D}_\alpha$ and c. It is impossible to say much about the existence or multiplicity of solutions without introducing some assumptions concerning the form of W. Given any set of initial data satisfying these conditions, we can calculate the $\mathbf{U}$ corresponding to $\boldsymbol{\omega}$, then use (26) and (27) to obtain the curve and $\mathbf{d}_\alpha$, these calculations being easy.

Now, using (23), (24), (25) and (34),

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times c\boldsymbol{\omega} = (b\boldsymbol{\omega} - \mathbf{r}')c.$$

Using (14), we then have

$$(b\boldsymbol{\omega} - \mathbf{r}')c + \mathbf{r}' \times \mathbf{G} + \mathbf{d}_\alpha \times \mathbf{H}^\alpha \equiv \mathbf{M} = \mathbf{b}.$$

Inspection of the left member shows, by reasoning like that used above,

$$\mathbf{M} = \mathbf{U}\mathbf{M}_0,$$

or

$$\mathbf{U}\mathbf{b} = \mathbf{b},$$

so that $\mathbf{b}$, the resultant moment acting on an end, has the property that

$$\mathbf{b} \parallel \boldsymbol{\omega}. \tag{36}$$

Here, the moment is calculated with respect to a particular point, bearing in mind the reasoning leading to (25).

This provides a general format for obtaining solutions for uniform states by algebraic methods. It would seem feasible to make a more detailed general study of the algebraic problem, but we leave the problem here.

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Абстракт—В большинстве нелинейных теорий стержней, некоторые предположения инверсии позволяют заменить определяющие дифференциальные уравнения—алгебраическими зависимостями. Для более общей абстрактивной теории стержней, представленной в работе, даются примеры как выполнить такое требование.